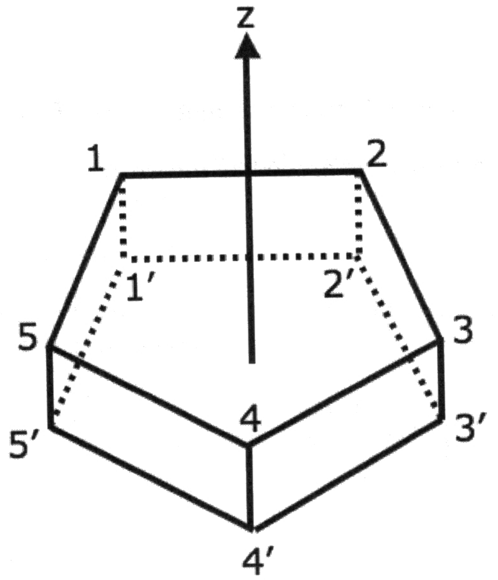
EXAM SYMMETRY IN PHYSICS

Wednesday, January 28, 2009, 9:00-12:00, room 18.-152

- Write your name and student number on the first page.
- Illegible writing will be graded as incorrect.
- Good luck!

Problem 1.

Consider the regular 3D pentagonal object in the Figure. The upper side (12345) is red, the lower side (1′2′3′4′5′) is blue, and the sides are white.



a. The symmetry group of this object we call G. Give the elements of G. Is G abelian?

b. Divide the elements of G in classes. How many irreducible representations (irreps) does G have? What are the dimensions of the irreps?

c. Construct the character table of G.

Suppose that the upper and lower side have the *same* color. Such an object has a different symmetry group that we call H.

d. How many elements does H have, and how are they related to the elements of G?

e. Construct the character table of H. *Hint:* The answer can be given independently from **c**, by representing the character table of G by a block.

Problem 2.

Consider an atom with one valence electron (we ignore spin). The energy levels in free space correspond to irreps of the rotation group SO(3). We apply a perturbation such that the symmetry is reduced to C_{4v}, the symmetry group of the square (rotations and reflections).

a. Give the elements of C_{4v} and divide them into conjugacy classes.

b. Derive the following character table, carefully explaining your reasoning:

C_{4v}	C_1	C_2	C_3	C_4	C_5
$D^{(1)}$	1	1	1	1	1
$D^{(2)}$	1	1	1	-1	-1
$D^{(3)}$	1	-1	1	-1	1
$D^{(4)}$	1	-1	1	1	-1
$D^{(5)}$	2	0	-2	0	0

c. For SO(3) the character of the conjugacy class labeled by the angle θ is

$$\chi^{(\ell)}(\theta) = \frac{\sin[(2\ell+1)\theta/2]}{\sin[\theta/2]} \,.$$

Derive this formula.

d. Calculate how the 7-fold degenerate $\ell = 3$ state is split by the C_{4v} perturbation. Give the new degeneracies. Draw a picture of the splitting.

e. We next apply a further perturbation that reduces the symmetry to C_{2v}, the symmetry group of the rectangle. Do you expect a further splitting? What are the remaining degeneracies?

Problem 3.

The Euclidean group E(2) contains all linear transformations in 2D Euclidean space that leave the length of all vectors invariant:

$$\begin{aligned} x'^1 &= x^1 \cos\theta - x^2 \sin\theta + b^1 \ , \\ x'^2 &= x^1 \sin\theta + x^2 \cos\theta + b^2 \ . \end{aligned}$$

The elements of E(2) are written as $g(\boldsymbol{b}, \theta)$.

a. Give the multiplication law, the unit element, and the inverse. How many parameters does E(2) have?

b. What are the subgroups of E(2)?

c. Show that a possible representation of the group is given by the three-dimensional matrix representation

$$g(\boldsymbol{b}, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & b^1 \\ \sin\theta & \cos\theta & b^2 \\ 0 & 0 & 1 \end{pmatrix}$$

acting on the vector $(x^1, x^2, 1)$.

d. Consider the Lie algebra of E(2). Derive the generators of the group.

e. Give the commutation relations and determine the structure constants of E(2).